Technical Appendix for Statistical Inference with Generalized Gini Indices of Inequality, Poverty and Welfare

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1 Preliminary Results

In this appendix we state formally the statistical results for all the indices and justify the approach to statistical inference using influence function based variance estimates. Before stating the results we first derive some preliminary results concerning the various pieces that are used in the indexes. In particular the S-Gini indices are all linear functionals of either the GLC, $\hat{G}$ or the LC $\hat{L}$. To derive the appropriate characterization of the limiting distribution of the indexes we show that the linear functional is Hadamard differentiable and derive appropriate weak convergence results for the GLC and LC after appropriate normalization. Similarly the E-Gini indexes are nonlinear functionals of either $p - \hat{L}(p)$ or $\hat{\mu}p - \hat{G}(p)$ so we derive the relevant weak convergence result for these processes and show that the functional is Hadamard differentiable. Our first result concerns the GLC and LC processes and gives a uniform consistency result, a weak convergence result and the pointwise influence function. We use $\mathcal{B}$ to refer to the Brownian Bridge process on $[0, 1]$ which is a mean zero Gaussian process with covariance kernel given by,

$$Cov(\mathcal{B}(p)\mathcal{B}(q)) = \min\{p, q\} - pq \tag{1}$$

Note that $\hat{G}$ and $\hat{L}$ are piecewise linear and continuous. Consequently we treat these objects as elements of $C[0, 1]$, the set of all continuous functions on $[0, 1]$. Define the Gaussian stochastic process, $\mathcal{G}$ on $[0, 1]$ to be such that for $p \in [0, 1]$,

$$\mathcal{G}(p) = -\int_0^p \frac{\mathcal{B}(t)}{f(Q(t))} dt$$

where $Q(t) = F^{-1}(t)$ is the quantile function at the point t in the support of the distribution F.

Lemma A1: *Given the assumptions on the distribution in the text (referred to as Assumption 1 in this appendix) we have that,*

(i) for the GLC $\sup |\hat{G}(p) - G(p)| \overset{a.s.}{\to} 0$ *and in the space* $C[0, 1]$,

$$\sqrt{N}(\hat{G} - G) \Rightarrow \mathcal{G}$$

with pointwise influence function given by,

$$\phi_i(p;\hat{G}) = (pQ(p) - G(p)) - 1(Y_i < Q(p))(Q(p) - Y_i)$$

(ii) for the LC we have that $\sup|\hat{L}(p) - L(p)| \overset{a.s.}{\to} 0$*and in the space* $C[0,1]$*,*

$$\sqrt{N}(\hat{L} - L) \Rightarrow \frac{\mathcal{G}}{\mu} - \frac{L}{\mu}\mathcal{G}(1) \equiv \mathcal{L}.$$

with pointwise influence function given by,

$$\phi_i(p;\hat{L}) = \frac{1}{\mu}\phi_i(p;\hat{G}) - \frac{L(p)}{\mu}(Y_i - \mu)$$

The results in Lemma A3 are not new and date back to at least Goldie (1977), who presented a full weak convergence result for the LC process under very weak conditions. Our proof of the results of Lemma A3 is somewhat simpler than that of Goldie (1977) since we take as a starting point the results concerning the quantile process which are stated below in Lemma in Lemma A2 which requires slightly stronger assumptions than required by the method of Goldie (1977). Other results concerning the empirical LC process include Gail and Gastwirth (1978) who derived an asymptotic distribution result for a single ordinate of the normalized LC and Csörgó (1983) who proved that the empirical LC process could be strongly approximated by a sequence of Gaussian processes which are equal in distribution to that given above. Note also that by standard arguments we have that $\hat{\mu} \overset{a.s.}{\to} \mu$, that $\sqrt{N}(\hat{\mu} - \mu)$ is asymptotically normally distributed with variance given by $E((Y_i - \mu)^2)$. Also, the influence function is given by $\phi_i(\hat{\mu}) = (Y_i - \mu)$. Indeed we get these results from Lemma A1(i) by noting that $\hat{\mu} = \hat{G}(1)$. Using this and the result in Lemma A1 we can straighforwardly derive results for $p - \hat{L}(p)$ and $\hat{\mu}.p - \hat{G}(p)$ which are used in the E-Gini indices. The influence functions for these objects are easily found as, $-\phi_i(p;\hat{L})$ and $p\phi_i(\mu) - \phi_i(p;G)$, respectively.

For the poverty indices we also need results for, $\hat{G}(p\hat{F}(z))$, $p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z))$, $\hat{F}(z)$ and $\hat{G}(\hat{F}(z))$ (with the latter two objects playing the roles of $\hat{\theta}_j$ in the text). It is well known that (see van der Vaart and Wellner (1996) for instance) that under our conditions $\hat{F}(z)$ is uniformly consistent and satisfies $\sqrt{N}(\hat{F} - F) \Rightarrow \mathcal{B}$ (in an appropriate function space) with a pointwise influence function given by $\phi_i(z;\hat{F}) = 1(Y_i \leq z) - F(z)$ so it remains to derive the influence function for, $\hat{G}(p\hat{F}(z))$ which also gives the result for $\hat{G}(\hat{F}(z))$ by setting $p = 1$. The following result shows that as a process defined over $[0,1]$, $\sqrt{N}(\hat{G}(p\hat{F}(z)) - G(pF(z)))$ converges to a Gaussian process with an influence function given in the statement of the result.

Lemma A2: *In $C[0,1]$ we have that $\sqrt{N}(\hat{G}(p\hat{F}(z)) - G(pF(z)))$ converges weakly to a Gaussian process with pointwise influence function given by,*

$$pQ(pF(z))\phi_i(z;\hat{F}) + \phi_i(pF(z);\hat{G})$$

This directly gives the influence function for $\hat{G}(\hat{F}(z))$ as a special case ($p = 1$) and allows us to obtain the influence function for $p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z))$ as,

$$p(Q(F(z))\phi_i(z;\hat{F}) + \phi_i(F(z);\hat{G}) - \Big(pQ(pF(z))\phi_i(z;\hat{F}) + \phi_i(pF(z);\hat{G})\Big)$$

The final set of preliminary results concerns the Hadamard differentiability of the maps $\int_0^1 (1-p)^{\delta-2}H(p)dp$ and $[\int_0^1 (H(p))^{\alpha}\, dp]^{\frac{1}{\alpha}}$ that play the role of T in the text and allow application of the functional delta method and allows calculation of the influence function of the indices.

Lemma A3:

(i) The functional, $\int_0^1 (1-p)^{\delta-2}H(p)dp$ is Hadamard differentiable with linear functional derivative given by,

$$T'_H(h) = \int_0^1 (1-p)^{\delta-2}h(p)dp$$

(ii) The functional $[\int_0^1 (H(p))^{\alpha}\, dp]^{\frac{1}{\alpha}}$ is Hadamard differentiable with linear functional derivative given by,

$$T'_H(h) = \left[\int_0^1 (H(p))^{\alpha}\, dp\right]^{\frac{1}{\alpha}-1} \int_0^1 (H(p))^{\alpha-1}\, h(p)dp$$

The first of these results follows immediately from the linearity of the functional $\int_0^1 (1-p)^{\delta-2} H(p)dp$. The second result follows from applying the functional methodology to the term $\int_0^1 (H(p))^\alpha dp$, which is a linear functional of the Hadamard differentiable map $(H(p))^\alpha$, and then applying the usual delta method to $[\int_0^1 (H(p))^\alpha dp]^{\frac{1}{\alpha}}$. We now have the pieces that are required to form the influence functions for the various indices.

1.0.1 S-Gini indices

The following result provides a characterization of the limiting distribution of the S-gini indices and also shows their consitency.

Proposition S1(A): *Given Assumption 1, the following results holds for a fixed value of δ, such that $1 < \delta < \infty$,*

(i) $\hat{I}_R^\delta \xrightarrow{p} I_R^\delta$ *and*

$$\sqrt{N}(\hat{I}_R^\delta - I_R^\delta) \Rightarrow -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\mathcal{L}(p)dp \sim N(0, V(\hat{I}_R^\delta))$$

(ii) $\hat{I}_R^\delta \xrightarrow{p} I_R^\delta$ *and*

$$\sqrt{N}(\hat{I}_A^\delta - I_A^\delta) \Rightarrow \mathcal{G}(1) - \delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\mathcal{G}(p)dp \sim N(0, V(\hat{I}_A^\delta))$$

(iii) $\hat{W}^\delta \xrightarrow{p} W^\delta$ *and*

$$\sqrt{N}(\hat{W}^\delta - W^\delta) \Rightarrow \delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\mathcal{G}(p)dp \sim N(0, V(\hat{W}^\delta)).$$

We denote by $\hat{V}(.)$ the estimate of the variance of the index using the average of the squared estimated influence function, so that for instance,

$$\hat{V}(\hat{W}^\delta) = \frac{1}{N}\sum_{i=1}^N \hat{\phi}_i(\hat{W}^\delta)^2$$

Proposition S1(B): *Given the conditions of Proposition 1(A),*

$\hat{V}(\hat{I}_R^\delta) \xrightarrow{p} V(\hat{I}_R^\delta)$,

$\hat{V}(\hat{I}_A^\delta) \xrightarrow{p} V(\hat{I}_A^\delta)$

$\hat{V}(\hat{W}^\delta) \xrightarrow{p} V(\hat{W}^\delta)$

1.0.2 E-Gini indices

Proposition E1(A) *Given Assumption 1, and assuming that I_R^α, I_A^α and W^α are all strictly positive then the following results hold,*

(i) $\hat{I}_R^\alpha \xrightarrow{p} I_R^\alpha$ *and,*

$$\begin{aligned}\sqrt{N}(\hat{I}_R^\alpha - I_R^\alpha) &\Rightarrow -2^\alpha (I_R^\alpha)^{1-\alpha}\int_0^1 (p-L(p))^{\alpha-1}\mathcal{L}(p)dp \\ &\sim N(0, V(\hat{I}_R^\alpha)),\end{aligned}$$

(ii) $\hat{I}_A^\alpha \xrightarrow{p} I_A^\alpha$ *and,*

$$\begin{aligned}\sqrt{N}(\hat{I}_A^\alpha - I_A^\alpha) &\Rightarrow 2^\alpha (I_A^\alpha)^{1-\alpha}\int_0^1 (\mu p-G(p))^{\alpha-1}(p\mathcal{G}(1)-\mathcal{G}(p))dp \\ &\sim N(0, V(\hat{I}_A^\alpha)),\end{aligned}$$

(iii) $\hat{W}^\alpha \xrightarrow{p} W^\alpha$ *and,*

$$\begin{aligned}\sqrt{N}(\hat{W}^\alpha - W^\alpha) &\Rightarrow 2\mathcal{G}(1) - 2^\alpha (I_A^\alpha)^{1-\alpha}\int_0^1 (\mu p-G(p))^{\alpha-1}(\mathcal{G}(p)-p\mathcal{G}(1))dp \\ &\sim N(0, V(\hat{W}^\alpha)).\end{aligned}$$

Again the variances can be expressed in terms of the influence functions which are,

$$\begin{aligned}\phi_i(I_A^\alpha) &= -2^\alpha (I_A^\alpha)^{1-\alpha}\int_0^1 (\mu p-G(p))^{\alpha-1}(\phi_i(p;G)-p(Y_i-\mu))dp \\ \phi_i(W^\alpha) &= 2(Y_i-\mu)+\phi_i(I_A^\alpha) \\ \phi_i(I_R^\alpha) &= \frac{1}{\mu}\phi_i(I_A^\alpha) - \frac{I_R^\alpha}{\hat{\mu}}(Y_i-\hat{\mu})\end{aligned}$$

Then the variances can be estimated using,

$$\begin{aligned}\hat{V}(\hat{W}^\alpha) &= \frac{1}{N}\sum_{i=1}^N \hat{\phi}_i(W^\alpha)^2 \\ \hat{V}(\hat{I}_A^\alpha) &= \frac{1}{N}\sum_{i=1}^N \hat{\phi}_i(I_A^\alpha)^2 \\ \hat{V}(\hat{I}_R^\alpha) &= \frac{1}{N}\sum_{i=1}^N \hat{\phi}_i(I_R^\alpha)^2\end{aligned}$$

where the $\hat{\phi}$ are ϕ with unknowns replaced by the relevant estimates. Appendix B indicates how these influence functions can be estimated for a particular sample.

Proposition E1(B): *Given Assumption 1,*

$\hat{V}(\hat{I}_R^\alpha) \xrightarrow{p} V(\hat{I}_R^\alpha)$,

$\hat{V}(\hat{I}_A^\alpha) \xrightarrow{p} V(\hat{I}_A^\alpha)$

$\hat{V}(\hat{W}^\alpha) \xrightarrow{p} V(\hat{W}^\alpha)$

1.0.3 Poverty Indices

Proposition S2(A): *Given Assumption 1, a fixed poverty level z, and $1 < \delta < \infty$, then* $\hat{P}^\delta \xrightarrow{p} P^\delta$ *and*

$$\begin{aligned}\sqrt{N}(\hat{P}^\delta - P^\delta) \quad &\Rightarrow \quad \mathcal{B}(F(z)) - \frac{\delta(\delta-1)}{z}\int_0^1 (1-p)^{\delta-2}\left\{Q(pF(z))p\mathcal{B}(F(z)) + \mathcal{G}(pF(z))\right\} dp \\ &\sim \quad N(0, V(\hat{P}^\delta))\end{aligned}$$

Proposition S2(B): *Given the conditions of Theorem 1* $\hat{V}(\hat{P}^\delta) \xrightarrow{p} V(\hat{P}^\delta)$.

We have separate result for the case where $\delta = 1$.

Proposition S3(A): *Given Assumption 1, a fixed poverty level z, and $\delta = 1$, then*

(i) $\hat{P}^\delta \xrightarrow{p} P^\delta$ *and*

$$\begin{aligned}\sqrt{N}(\hat{P}^\delta - P^\delta) \quad &\Rightarrow \quad N\left(0, \frac{1}{z^2}\left\{E\left[(z-Y_i)^2.1(Y_i \le z)\right] - E\left[(z-Y_i).1(Y_i \le z))\right]^2\right\}\right) \\ &\equiv \quad N(0, V(\hat{P}^\delta))\end{aligned}$$

(ii)

$$\hat{V}(\hat{P}^\delta) = \frac{1}{Nz^2}\sum_{i=1}^{N}(z-Y_i)^2.1(Y_i \le z)) - (\hat{P}^\delta)^2 \xrightarrow{p} V(\hat{P}^\delta)$$

Proposition S3(B): *Given the conditions of Theorem 1 then for* $\delta = 1$, $\hat{V}(\hat{P}^\delta) \xrightarrow{p} V(\hat{P}^\delta)$.

Finally for the Poverty index based on the E-gini index we have the following result.

Proposition E2(A): *Given Assumption 1, a fixed poverty level* z*,and* $1 < \alpha < \infty$*, then* $\hat{P}^{\alpha} \xrightarrow{p} P^{\alpha}$ *and*

$$\begin{aligned}\sqrt{N}(\hat{P}^{\alpha} - P^{\alpha}) &\Rightarrow \mathcal{B}(F(z)) - \frac{1}{z}\mathcal{G}(F(z)) - 2\mathcal{B}(F(z)) \\ &\quad + \frac{1}{z}[T_E^z]^{\frac{1}{\alpha}-1}\int_0^1 (pG(F(z)) - G(pF(z)))^{\alpha-1}\Upsilon(p, F(z))dp) \\ &\sim N(0, V(\hat{P}^{\alpha})).\end{aligned}$$

where,

$$T_E^z = \int_0^1 (pG(F(z)) - G(pF(z)))^{\alpha} dp$$

and,

$$\Upsilon(p, F(z)) = p\,(z - Q(pF(z)))\,\mathcal{B}(F(z)) + p\mathcal{G}(F(z)) - \mathcal{G}(pF(z)).$$

Proposition E2(B): *Given the conditions of Theorem 1* $\hat{V}(\hat{P}^{\alpha}) \xrightarrow{p} V(\hat{P}^{\alpha})$.

2 Proofs

Note that we use a result concerning the quantile process.

Lemma A4: *Given the Assumption 1, the following results hold,*

(i) $\sup|\hat{F}(y) - F(y)| \stackrel{a.s.}{\to} 0$ *and in the space* $D([y_l, y_u])$ *(the space of cadlag functions),*

$$\sqrt{N}(\hat{F} - F) \Rightarrow \mathcal{B}\circ F$$

which is a Brownian Bridge with covariance between the process at x *and* z *given by the expression in (1) with* $p = F(x)$ *and* $q = F(z)$*. The pointwise influence function is given by,*

$$\phi_i(x; \hat{Q}) = 1(Y_i \leq x) - F(x)$$

(ii) $\sup|\hat{Q}(p) - Q(p)| \stackrel{a.s.}{\to} 0$ *and in* $l^{\infty}(0,1)$ *(the space of uniformly bounded real functions on* $(0,1)$*) we have,*

$$\sqrt{N}(\hat{Q} - Q) \Rightarrow -\frac{\mathcal{B}}{f(Q)}$$

where the the pointwise influence function is given by,

$$\phi_i(p; \hat{Q}) = 1(Y_i < Q(p))\frac{p-1}{f(Q(p))} + 1(Y_i > Q(p))\frac{p}{f(Q(p))}$$

The result in (i) are well known results for the empirical distribution function, see van der Vaart and Wellner (1996). The results in (ii) combine the strong uniform convergence result contained in Corollary 1.4.1 of Csorgo (1983), a weak convergence result such as shown in Van der Vaart and Wellner (1996, page 387) and the influence function is derived in Huber (1981, p. 56). See Csorgo (1983) for more details on quantile processes.

Proof of Lemma A1: (i) Note that ,

$$\hat{G}(p) - G(p) = \int_0^p (\hat{Q}(t) - Q(t))dt.$$

Consequently we have that,

$$\begin{aligned}
\sup_p |\hat{G}(p) - G(p)| &= \sup_p |\int_0^p (\hat{Q}(t) - Q(t))dt| \\
&\leq \sup_p \int_0^p \sup_{t\in[0,p]} |\hat{Q}(t) - Q(t)|dt \\
&\leq \sup_{t\in[0,1]} |\hat{Q}(t) - Q(t)| \sup_p \int_0^p dt \\
&= \sup_{t\in[0,1]} |\hat{Q}(t) - Q(t)| \\
&= o_p(1)
\end{aligned}$$

using the result in Lemma A4(i). For the next result note that $G = T(Q)$ is a linear functional of Q such that,

$$T(Q)(p) = \int_0^p Q(t)dt$$

and is therefore Hadamard differentiable at Q tangentially to $C[0,1]$ with Hadamard derivative T' with

$$T'(\tilde{Q})(p) = \int_0^p \tilde{Q}(t)dt$$

for $\tilde{Q} \in C[0,1]$. Consequently,

$$\sqrt{N}(\hat{G} - G) \Rightarrow \mathcal{G}$$

in $l^\infty(0,1)$ where,

$$\mathcal{G}(p) = -\int_0^p \frac{\mathcal{B}(t)}{f(Q(t))}dt.$$

To get the influence function we have (using a change of variable from t to y) that,

$$\begin{aligned}
\phi_i(p;G) &= \int_0^p \phi_i(t;Q)dt \\
&= \int_0^p \left(1(Y_i < Q(t))\frac{t-1}{f(Q(t))} + 1(Y_i > Q(t))\frac{t}{f(Q(t))}\right) dt \\
&= -\int_{y_l}^{y_p} 1(Y_i < y)\,(1 - F(y))\,dy + \int_{y_l}^{y_p} 1(Y_i > y)F(y)dy \\
&= -1(Y_i < y_p)\left(\int_{Y_i}^{y_p} (1 - F(y))\,dy - \int_{y_l}^{Y_i} F(y)dy\right) + 1(Y_i > y_p)\int_{y_l}^{y_p} F(y)dy \\
&= (py_p - G(p)) - 1(Y_i < y_p)(y_p - Y_i)
\end{aligned}$$

after using integration by parts to show that,

$$\begin{aligned}\int_{y_l}^{y_p} F(y)dy &= yF(y)|_{y_l}^{y_p} - \int_{y_l}^{y_p} yf(y)dy \\ &= y_pF(y_p) - G(p)\end{aligned}$$

Then take the definition of $G(p)$ and the facts that $F(y_p) = p$ and $y_p = Q(p)$ and the result follows.(ii) For the LC we have that since $\hat{L}(p) = \hat{G}(p)/\hat{\mu}$ then,

$$\begin{aligned}\sup_p |\hat{L}(p) - L(p)| &\leq \frac{1}{\hat{\mu}} \sup_p |\hat{G}(p) - G(p)| + \frac{L(p)}{\hat{\mu}}|\mu - \hat{\mu}| \\ &\xrightarrow{p} 0\end{aligned}$$

by result (i) and $\hat{\mu} \xrightarrow{p} \mu$ with $0 < \mu < \infty$ which follows from Assumption 1. Similarly,

$$\begin{aligned}\sqrt{N}(\hat{L} - L) &= \frac{1}{\hat{\mu}}\sqrt{N}(\hat{G} - G) - \frac{L}{\hat{\mu}}\sqrt{N}(\hat{\mu} - \mu) \\ &\Rightarrow \frac{\mathcal{G}}{\mu} - \frac{L}{\mu}\mathcal{G}(1)\end{aligned}$$

using the Slutsky Theorem (ST) the Continuous Mapping Theorem (CMT) and the fact that,

$$\sqrt{N}(\hat{\mu} - \mu) \Rightarrow \mathcal{G}(1)$$

since $\hat{\mu} = \hat{G}(1)$. To get the influence function we use the fact that,

$$\hat{L}(p) = \frac{\hat{G}(p)}{\hat{G}(1)}$$

so that by the product rule for differentiation,

$$\begin{aligned}\phi_i(p; L) &= \frac{1}{G(1)}\phi_i(p; G) - \frac{G(p)}{G(1)^2}\phi_i(1; G) \\ &= \frac{1}{\mu}\phi_i(p; G) - \frac{L(p)}{\mu}(Y_i - \mu)\end{aligned}$$

using the fact that $G(1) = \mu$.**Q.E.D.**

Proof of Lemma A2: To do this we first write,

$$\begin{aligned}\sqrt{N}(\hat{G}(p\hat{F}(z)) - G(pF(z))) = &\sqrt{N}(\hat{G}(pF(z)) - G(pF(z))) \\ &+ \sqrt{N}(G(p\hat{F}(z)) - G(pF(z))) \\ &+ \sqrt{N}\left((\hat{G}(p\hat{F}(z)) - G(p\hat{F}(z))) - (\hat{G}(pF(z)) - G(pF(z)))\right)\end{aligned}$$

For the last term we have,

$$\begin{aligned}
& \sup_p \left| \sqrt{N} \left((\hat{G}(p\hat{F}(z)) - G(p(\hat{F}(z))) - (\hat{G}(pF(z)) - G(pF(z))) \right) \right| \\
= & \sup_p \left| \sqrt{N} \int_{pF(z)}^{p\hat{F}(z)} (\hat{Q}(t) - Q(t)) dt \right| \\
\leq & \sup_p \left| \hat{Q}(p) - Q(p)) \right| \sqrt{N} \sup_p |p\hat{F}(z) - pF(z)| \\
\leq & \sup_p \left| \hat{Q}(p) - Q(p)) \right| \sqrt{N} |\hat{F}(z) - F(z)| \\
= & o_p(1)
\end{aligned}$$

by Corollary 1.4.1 of Csorgo (1983) which implies that $\sup_p \left| \hat{Q}(p) - Q(p)) \right| \xrightarrow{p} 0$ and the usual Central Limit Theorem applied to $\sqrt{N}|\hat{F}(z) - F(z)|$. By a simple change of variable we have that the first term converges weakly to Gaussian process by Lemma A1, and that the influence function will take the form $\phi_i(pF(z); \hat{G})$. For the second term we note that$\sqrt{N}(p\hat{F}(z) - pF(z))$ converges to a Gaussian process and has pointwise influence function given by $p\phi_i(z; \hat{F})$ while, on the interval $(0, 1)$ the function $G(t)$ has a derivative $Q(t)$ which is continuous on $[0, 1]$ and hence bounded and uniformly continuous on $(0, 1)$ then since for $p \in (0, 1)$, $0 < pF(z) < 1$ so that by Lemma 3.9.25 of Van der Vaart and Wellner (1996) implies that the map $G(t)$ is Hadamard differentiable so that $\sqrt{N}(G(p\hat{F}(z)) - G(pF(z)))$ converges weakly to a Gaussian process that had pointwise influence function given by, $Q(pF(z))p\phi_i(z; \hat{F})$. Therefore the incluence function for $\sqrt{N}(\hat{G}(p\hat{F}(z)) - G(pF(z)))$, is given by,

$$\phi_i(pF(z); \hat{G}) + Q(pF(z))p\phi_i(z; \hat{F}).$$

Q.E.D.

Proof of Lemma A3: Since $T(H) = \int_0^1 (1-p)^{\delta-2} H(p) dp$ is a linear functional of $H(p)$ it is trivially Hadarmard differentiable and for $h_n(p) \to h(p)$, $t_n \to 0$ with $H(p) + t_n h_n(p)$, $h(p)$ and $H(p)$ in the class of continuous functions,

$$\begin{aligned}
\frac{T(H + t_n h_n) - T(H)}{t_n} &= \int_0^1 (1-p)^{\delta-2} h_n(p)) dp \\
&\to \int_0^1 (1-p)^{\delta-2} h(p)) dp
\end{aligned}$$

so the result in (i) follows.

For $[\int_0^1 (H(p))^\alpha \, dp]^{1/\alpha}$ it suffices to show that the $(H(p))^\alpha$ defined as a mapping from $(0,1) \to R$ for some finite Δ, is Hadamard differentiable. Note that $H(p) \in [0,\Delta]$ for some finite Δ. Note that the function,

$$g_1(x) = x^\alpha \text{ for } x \in [0,\Delta] \text{ for some finite } \Delta$$

is differentiable at $x \in [0,\Delta]$ with derivative given by $\alpha x^{\alpha-1}$ which is uniformly continuous on $[0,\Delta]$ and hence uniformly continuous and bounded on $(0,\Delta)$. Since for $p \in (0,1)$ we have that, $0 < H(p) < \Delta$ then by Lemma 3.9.25 of Van der Vaart and Wellner (1996) the mapping $(H(p))^\alpha$ is Hadamard differentiable with linear functional derivative

$$\alpha \, (H(p))^{\alpha-1} \, h(p)$$

for h in the class of continuous functions on $[0,1]$. Then $\int_0^1 (H(p))^\alpha \, dp$, being a linear functional of a Hadamard differentiable function, is trivially Hadamard differentiable with linear functional derivative,

$$\int_0^1 \alpha \, (H(p))^{\alpha-1} \, h(p) dp$$

Since the latter is simply a real valued object (rather than a function) and since the function,

$$g_2(x) = (x)^{1/\alpha}$$

is differentaible with derivative,

$$g_2'(x) = (1/\alpha)(x)^{(1/\alpha)-1}$$

we have that $[\int_0^1 (H(p))^\alpha \, dp]^{\frac{1}{\alpha}}$ is Hadamard differentiable with linear functional derivative,

$$(\int_0^1 (H(p))^\alpha \, dp)^{(1/\alpha)-1} \int_0^1 (H(p))^{\alpha-1} \, h(p) dp.$$

Q.E.D.

Lemma A5: *For an influence curve* ϕ_i*with estimate* $\hat{\phi}_i$*with*

$$|\hat{\phi}_i - \phi_i| \leq \Delta_1 \hat{A}_1 + \Delta_{2i}\hat{A}_2 + \Delta_3 \hat{A}_{3i} + \Delta_{4i}\hat{A}_{4i}$$

then sufficient conditions for,

$$\frac{1}{N}\sum_{i=1}^{N}\hat{\phi}_i^2 - \frac{1}{N}\sum_{i=1}^{N}\phi_i^2 = o_p(1)$$

are that the following hold,

(i) $\hat{A}_1 \xrightarrow{p} 0$ *and* $|\Delta_1|$ *is stochastically bounded;;*

(ii) $\hat{A}_2 \xrightarrow{p} 0$ *and*$\left|\frac{1}{N}\sum_{i=1}^{N}\Delta_{2i}^2\right|$ *is stochastically bounded;*

(iii) $\frac{1}{N}\sum_{i=1}^{N}\hat{A}_{3i}^2 \xrightarrow{p} 0$ *and* $|\Delta_3|$ *is stochastically bounded;*

(iv) $\frac{1}{N}\sum_{i=1}^{N}\hat{A}_{4i}^2 \xrightarrow{p} 0$ *and* $\frac{1}{N}\sum_{i=1}^{N}\Delta_{4i}^4$ *is stochastically bounded;*

(v)$\frac{1}{N}\sum_{i=1}^{N}\phi_i^2$ *is stochastically bounded.*

Proof: This follows from the fact that,

$$\begin{aligned}\left|\frac{1}{N}\sum_{i=1}^{N}\hat{\phi}_i^2 - \frac{1}{N}\sum_{i=1}^{N}\phi_i^2\right| &= \left|\frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}_i - \phi_i)^2 + \frac{2}{N}\sum_{i=1}^{N}\phi_i(\hat{\phi}_i - \phi_i)\right| \\ &\leq \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}_i - \phi_i)^2 \\ &\quad +2\left(\frac{1}{N}\sum_{i=1}^{N}\phi_i^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}_i - \phi_i)^2\right)^{1/2}\end{aligned}$$

and the conditions given with repeated use of the Cauchy Schwarz inequality. **Q.E.D.**

Proof of Proposition S1(A): Each of these results follow from Lemma A1 and Lemma A3. In particular for (i),

$$\begin{aligned}|\hat{I}_R^{\delta} - I_R^{\delta}| &\leq \delta \sup_p |\hat{L}(p) - L(p)| \int_0^1 (\delta - 1)(1-p)^{\delta-2}dp \\ &= \delta \sup_p |\hat{L}(p) - L(p)| = o_p(1)\end{aligned}$$

using Lemma A1(ii). Also,

$$\sqrt{N}(\hat{I}_R^\delta - I_R^\delta) \Rightarrow -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\mathcal{L}(p)dp \sim N(0, V(\hat{I}_R^\delta))$$

by Lemma A3 and the result in Lemma A3(ii). Exactly analogous arguments yield the results in (ii) and (iii) using the result contained in Lemma A1. **Q.E.D.**

Proof of Proposition S1(B): We verify the conditions of Lemma A5. Consider first

$$\hat{\phi}_i(\hat{W}^\delta) = \delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\hat{\phi}_i(p;\hat{G})dp$$

and write,

$$\hat{\phi}_i(\hat{W}^\delta) - \phi_i(\hat{W}^\delta) = \sum_{j=1}^4 T_j$$

where,

$$\begin{aligned}
T_1 &= \delta(\delta-1)\int_0^1 (1-p)^{\delta-2}p(\hat{Q}(p) - Q(p))dp \\
T_2 &= -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}(\hat{G}(p) - G(p))dp \\
T_3 &= -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}(1(Y_i < \hat{Q}(p))\hat{Q}(p) - 1(Y_i < Q(p))Q(p))dp \\
T_4 &= Y_i\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}(1(Y_i < \hat{Q}(p)) - 1(Y_i < Q(p)))dp
\end{aligned}$$

For the first term we have,

$$\begin{aligned}
|T_1| &\leq \sup_p |\hat{Q}(p) - Q(p)|\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}pdp \\
&= \sup_p |\hat{Q}(p) - Q(p)| \\
&= o_p(1)
\end{aligned}$$

using Lemma A4(ii). Similarly using Lemma A1(i),

$$\begin{aligned}
|T_2| &\leq \sup_p |\hat{G}(p) - G(p)|\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}pdp \\
&= o_p(1)
\end{aligned}$$

Next,

$$
\begin{aligned}
|T_3| &= \delta(\delta-1)\left|\int_{\hat{F}(Y_i)}^{1}(1-p)^{\delta-2}\hat{Q}(p)dp - \int_{F(Y_i)}^{1}(1-p)^{\delta-2}Q(p)dp\right| \\
&\leq \delta(\delta-1)\left|\int_{\hat{F}(Y_i)}^{1}(1-p)^{\delta-2}\hat{Q}(p)dp - \int_{\hat{F}(Y_i)}^{1}(1-p)^{\delta-2}Q(p)dp\right| \\
&\quad +\delta(\delta-1)\left|\int_{\hat{F}(Y_i)}^{1}(1-p)^{\delta-2}Q(p)dp - \int_{F(Y_i)}^{1}(1-p)^{\delta-2}Q(p)dp\right| \\
&\leq \sup_p |\hat{Q}(p)-Q(p)|\delta(\delta-1)\int_0^1(1-p)^{\delta-2}dp \\
&\quad +\left|\delta(\delta-1)\int_{\hat{F}(Y_i)}^{F(Y_i)}(1-p)^{\delta-2}Q(p)dp\right| \\
&\leq \sup_p |\hat{Q}(p)-Q(p)|\delta + \sup_p |Q(p)| \sup_y |(1-\hat{F}(y))^{\delta-1} - (1-F(y))^{\delta-1}| \\
&= o_p(1)
\end{aligned}
$$

where the last line follows from Assumption 1, and the results in Lemma A4 using the fact that the function $g(x) = x^{\delta-1}$ is uniformly continuous on $[0, 1]$ for $\delta > 1$. Finally,

$$
\begin{aligned}
|T_4| &\leq Y_i\delta(\delta-1)\left|\int_0^1(1-p)^{\delta-2}(1(Y_i<\hat{Q}(p)) - 1(Y_i<Q(p)))dp\right| \\
&\leq Y_i\delta\left|\int_{\hat{F}(Y_i)}^{F(Y_i)}(\delta-1)(1-p)^{\delta-2}dp\right| \\
&\leq Y_i \sup_y |(1-\hat{F}(y))^{\delta-1} - (1-F(y))^{\delta-1}|.
\end{aligned}
$$

This term the satisfies (ii) of Lemma A5 so that because all the other terms satisfy (i) of Lemma A5 we have the result in (iii). The results in (i) and (ii) follow similarly. For (ii),

$$
|\hat{\phi}_i(I_A^\delta) - \phi_i(I_A^\delta)| \leq |\hat{\mu}-\mu| + |\hat{\phi}_i(W^\delta) - \phi_i(W^\delta)|
$$

so that the result in (i) follows from (iii) and Lemma A5 using the fact that $\hat{\mu}-\mu = o_p(1)$. For (i),

$$
\begin{aligned}
|\hat{\phi}_i(\hat{I}_R^\delta) - \phi_i(I_R^\delta)| &\leq \frac{1}{\hat{\mu}}|\hat{\phi}_i(I_A^\delta) - \phi_i(I_A^\delta)| + |\phi_i(I_A^\delta)|\left|\frac{1}{\hat{\mu}} - \frac{1}{\mu}\right| \\
&\quad + \left|\frac{\hat{I}_R^\delta}{\hat{\mu}}\right||\hat{\mu}-\mu| + |Y_i-\mu|\left|\frac{\hat{I}_R^\delta}{\hat{\mu}} - \frac{I_R^\delta}{\mu}\right|
\end{aligned}
$$

so that the result follows from (ii) and Lemma A5 using $\hat{\mu} - \mu = o_p(1)$ the result of Proposition S1(A)(i) and Assumption 1 which implies that $E(|Y_i - \mu|^2) < \infty$. **Q.E.D.**

Proof of Proposition E1(A): Consistency of all three indices follows simply from the CMT. More specifically consider first the estimator,

$$\hat{I}_R^\alpha = 2[\int_0^1 (p - \hat{L}(p))^\alpha dp]^{\frac{1}{\alpha}}$$

and consider the component $(p - \hat{L}(p))^\alpha$. Note that $p - \hat{L}(p) \in [0,1]$. Now consider the function $g(z) = z^\alpha$ for $z \in [0,1]$ with $\alpha \geq 1$ fixed. It is easy to show that for $z_1, z_2 \in [0,1]$, $|g(z_1) - g(z_2)| \leq \alpha |z_1 - z_2|$ so that,

$$|(p - \hat{L}(p))^\alpha - (p - L(p))^\alpha| \leq \alpha |\hat{L}(p) - L(p)|$$

so that,

$$\begin{aligned}\left|\int_0^1 (p - \hat{L}(p))^\alpha dp - \int_0^1 (p - L(p))^\alpha dp\right| &\leq \int_0^1 |(p - \hat{L}(p))^\alpha - (p - L(p))^\alpha| dp \\ &\leq \alpha \sup |\hat{L}(p) - L(p)| \\ &= o_p(1)\end{aligned}$$

using Lemma A1(ii) so that

$$\int_0^1 (p - \hat{L}(p))^\alpha dp \xrightarrow{p} \int_0^1 (p - L(p))^\alpha dp$$

Then it follows by the CMT that

$$\hat{I}_R^\alpha = 2[\int_0^1 (p - \hat{L}(p))^\alpha dp]^{\frac{1}{\alpha}} \xrightarrow{p} 2[\int_0^1 (p - L(p))^\alpha dp]^{\frac{1}{\alpha}} = I_R^\alpha$$

For the other indices we use the CMT and ST and the facts that,

$$\begin{aligned}\hat{I}_A^\alpha &= \hat{\mu}\hat{I}_R^\alpha \xrightarrow{p} \mu I_R^\alpha = I_A^\alpha \\ \hat{W}^\alpha &= 2\hat{\mu} - \hat{I}_A^\alpha \xrightarrow{p} 2\mu - I_A^\alpha = W^\alpha\end{aligned}$$

In order to derive the limiting distribution results we use the result of Lemma A1(ii) which implies that,

$$\begin{aligned}\sqrt{N}\left(\left(p - \hat{L}(p)\right) - (p - L(p))\right) &= -\sqrt{N}(\hat{L}(p) - L(p)) \\ &\Rightarrow -\mathcal{L}(p)\end{aligned}$$

Therefore using Lemma A3,

$$\begin{aligned}\sqrt{N}(\hat{I}_R^\alpha - I_R^\alpha) &\Rightarrow -2A^{(1/\alpha)-1}\int_0^1 (p - L(p))^{\alpha-1}\mathcal{L}(p)dp \\ A &= \int_0^1 (p - L(p))^\alpha dp\end{aligned}$$

The result in (i) follows upon noting that

$$2A^{(1/\alpha)-1} = 2^\alpha 2^{1-\alpha}(A^{1/\alpha})^{1-\alpha} = 2^\alpha(2A^{1/\alpha})^{1-\alpha} = 2^\alpha(I_R^\alpha)^{1-\alpha}$$

For (ii) we note that

$$\begin{aligned}\sqrt{N}(\hat{I}_A^\alpha - I_A^\alpha) &= \hat{\mu}\sqrt{N}(\hat{I}_R^\alpha - I_R^\alpha) + I_R^\alpha\sqrt{N}(\hat{\mu} - \mu) \\ &\Rightarrow -\mu 2A^{(1/\alpha)-1}\int_0^1 (p - L(p))^{\alpha-1}\mathcal{L}(p)dp + I_R^\alpha\mathcal{G}(1) \\ &\equiv 2^\alpha(I_A^\alpha)^{1-\alpha}\int_0^1 (\mu p - G(p))^{\alpha-1}(p\mathcal{G}(1) - \mathcal{G}(p))dp\end{aligned}$$

using the definition of $\mathcal{L}$ after some manipulations. The result for

$$\hat{W}^\alpha = 2\hat{\mu} - \hat{I}_A^\alpha$$

follows similarly. **Q.E.D.**

Proof of Proposition E1(B): Write,

$$\begin{aligned}|\hat{\phi}_i(I_A^\alpha) - \phi_i(I_A^\alpha)| &\leq 2^\alpha(\hat{I}_A^\alpha)^{1-\alpha}\sum_{j=1}^{7}|T_j| \\ &\quad + 2^\alpha((\hat{I}_A^\alpha)^{1-\alpha} - (\hat{I}_A^\alpha)^{1-\alpha})\phi_i^\alpha\end{aligned}$$

where,

$$\begin{aligned}
T_1 &= \int_0^1 \Big((\hat{\mu}p - \hat{G}(p))^{\alpha-1} p\hat{Q}(p) - (\mu p - G(p))^{\alpha-1} pQ(p) \Big)\, dp \\
T_2 &= \int_0^1 \Big((\hat{\mu}p - \hat{G}(p))^{\alpha-1} \hat{G}(p) - (\mu p - G(p))^{\alpha-1} G(p) \Big)\, dp \\
T_3 &= \int_0^1 \Big((\hat{\mu}p - \hat{G}(p))^{\alpha-1} \hat{Q}(p) - (\mu p - G(p))^{\alpha-1} Q(p) \Big)\, 1(Y_i < \hat{Q}(p)) dp \\
T_4 &= \int_0^1 \Big((\mu p - G(p))^{\alpha-1} Q(p)(1(Y_i < \hat{Q}(p)) - 1(Y_i < Q(p))) \Big)\, dp \\
T_5 &= Y_i \int_0^1 \Big((\hat{\mu}p - \hat{G}(p))^{\alpha-1} - (\mu p - G(p))^{\alpha-1} \Big)\, 1(Y_i < \hat{Q}(p)) dp \\
T_6 &= Y_i \int_0^1 \Big((\mu p - G(p))^{\alpha-1} (1(Y_i < \hat{Q}(p)) - 1(Y_i < Q(p))) \Big)\, dp \\
T_7 &= \int_0^1 \Big((\hat{\mu}p - \hat{G}(p))^{\alpha-1} p\hat{\mu} - (\mu p - G(p))^{\alpha-1} p\mu \Big)\, dp \\
\phi_i^\alpha &= \int_0^1 (\mu p - G(p))^{\alpha-1} (\phi_i(p; G) - p(Y_i - \mu)) dp
\end{aligned}$$

Because of the fact that $2^\alpha(\hat{I}_A^\alpha)^{1-\alpha}$ is stochastically bounded we must show that $|T_j| = o_p(1)$ so that the conditions of Lemma A5 are satisfied. To deal with T_1 use the inequality,

$$|\hat{a}\hat{b} - ab| = |\hat{a} - a||\hat{b} - b| + |a||\hat{b} - b| + |b||\hat{a} - a|$$

and the following facts,

$$\begin{aligned}
\sup_p |Q(p)| &< \infty \\
\sup_p |(\mu p - G(p))^{\alpha-1}| &\leq \mu < \infty \\
\sup_p |p(\hat{Q}(p) - Q(p))| &\leq \sup_p |(\hat{Q}(p) - Q(p)| = o_p(1)
\end{aligned}$$

which follow from Assumption 1 and Lemma A4(ii) and the fact that,

$$\sup_p |(\hat{\mu}p - \hat{G}(p))^{\alpha-1} - (\mu p - G(p))^{\alpha-1}| = o_p(1)$$

This last result follows from Lemma A1(i) and the fact that the function $g(x) = x^{\alpha-1}$ is such that for $1 < \alpha \leq 2$ and $x', x'' \in [0, 1]$,

$$|g(x') - g(x'')| \leq |x' - x''|^{\alpha-1}$$

while for $\alpha > 2$,

$$|g(x') - g(x'')| \leq (\alpha - 1)|x' - x''|$$

Then combining these results we have that,

$$
\begin{aligned}
|T_1| &\leq \int_0^1 \sup_p \left|(\hat{\mu}p - \hat{G}(p))^{\alpha-1}p\hat{Q}(p) - (\mu p - G(p))^{\alpha-1}pQ(p)\right| dp \\
&\leq \sup_p \left|(\hat{\mu}p - \hat{G}(p))^{\alpha-1}p\hat{Q}(p) - (\mu p - G(p))^{\alpha-1}pQ(p)\right| \\
&= o_p(1)
\end{aligned}
$$

The same arguments apply to the term T_2 and T_7 using the results in Lemma A1. For T_3 and T_5 the same arguments apply once we note that,

$$\sup_p 1(Y_i < \hat{Q}(p)) \leq 1.$$

Next $|T_5| = Y_i o_p(1)$ follows using the fact that,

$$\sup_p |(\mu p - G(p))^{\alpha-1}Q(p)| < \infty$$

and,

$$
\begin{aligned}
|\int_0^1 \Big(1(Y_i < \hat{Q}(p)) - 1(Y_i < Q(p))\Big) dp| &= |\hat{F}(Y_i) - F(Y_i)| \\
&\leq \sup_y |\hat{F}(y) - F(y)| \\
&= o_p(1)
\end{aligned}
$$

which follows from Lemma A4(i). Finally by Proposition E1(A)(ii) we have that, $(\hat{I}_A^\alpha)^{1-\alpha} - (\hat{I}_A^\alpha)^{1-\alpha} = o_p(1)$ and since

$$|\phi_i^\alpha| \leq C_1 + C_2 Y_i$$

we have that condition (ii) of Lemma A5 is satisfied by the term $2^\alpha((\hat{I}_A^\alpha)^{1-\alpha} - (\hat{I}_A^\alpha)^{1-\alpha})\phi_i^\alpha$. The results for the other indices follow in a manner that is similar to the proof of (i) and (ii) of Proposition E1(B). **Q.E.D.**

Proof of Proposition S2(A): Given,

$$\hat{P}^\delta = \hat{F}(z) - \frac{1}{z}\delta(\delta - 1)\int_0^1 (1-p)^{\delta-2}\hat{G}(p\hat{F}(z))dp$$

and the fact that by Lemma A4(i),

$$\sqrt{N}(\hat{F}(z) - F(z)) \Rightarrow \mathcal{B}(F(z))$$

and the fact that by Lemma A2 and Lemma A3(i),

$$\sqrt{N}\int_0^1 (1-p)^{\delta-2}(\hat{G}(p\hat{F}(z))-G(pF(z)))dp \Rightarrow \int_0^1 (1-p)^{\delta-2}\left\{Q(pF(z))p\mathcal{B}(F(z)) + \mathcal{G}(pF(z))\right\} dp.$$

The result then follows similarly to the proof of Proposition S1(A).**Q.E.D.**

Proof of Proposition S2(B): Note that,

$$|\hat{\phi}_i(P^\delta) - \phi_i(P^\delta)| \leq \frac{1}{z}\sum_{j=1}^{5} T_j$$

where,

$$\begin{aligned}
T_1 &= \left|1(Y_i \leq z)\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}p(\hat{Q}(p\hat{F}(z)) - Q(pF(z)))dp\right| \\
T_2 &= \left|\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}p(\hat{F}(z)\hat{Q}(p\hat{F}(z)) - F(z)Q(pF(z)))dp\right| \\
T_3 &= \left|\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}(\hat{G}(p\hat{F}(z)) - G(pF(z)))dp\right| \\
T_4 &= \left|\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}1(Y_i < \hat{Q}(p\hat{F}(z)))(\hat{Q}(p\hat{F}(z)) - Q(pF(z)))dp\right| \\
T_5 &= \left|\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}Q(pF(z))(1(Y_i < \hat{Q}(p\hat{F}(z))) - 1(Y_i < Q(pF(z))))dp\right| \\
T_6 &= \left|Y_i\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\left(1(Y_i < \hat{Q}(pF(z))) - 1(Y_i < Q(pF(z))\right) dp\right|
\end{aligned}$$

For the first term note that,

$$\hat{Q}(p\hat{F}(z)) - Q(pF(z)) = (\hat{Q}(p\hat{F}(z)) - Q(p\hat{F}(z))) - (Q(p\hat{F}(z)) - Q(pF(z)))$$

so that,

$$\sup_p |\hat{Q}(p\hat{F}(z)) - Q(pF(z))| \leq \sup_p |\hat{Q}(p) - Q(p)| + \sup_p |Q(p\hat{F}(z)) - Q(pF(z))|$$

By Lemma A4(ii) we have that,

$$\sup_p |\hat{Q}(p) - Q(p)| = o_p(1)$$

Since Q is uniformly continuous on $[0,1]$ then for any $\epsilon > 0$ there is a $\delta > 0$ such that if $p', p'' \in [0,1]$ with $|p' - p''| < \delta$ then $|Q(p') - Q(p'')| < \epsilon$. By Lemma A4(i) we have that

$\hat{F}(z) \xrightarrow{p} F(z)$ so that with probability approaching 1, $|p\hat{F}(z) - pF(z)| < \delta$ which implies that with probability approaching 1,

$$\sup_p |Q(p\hat{F}(z)) - Q(pF(z))| < \epsilon$$

and since ϵ is arbitrary we have that,

$$\sup_p |Q(p\hat{F}(z)) - Q(pF(z))| \xrightarrow{p} 0.$$

Given these facts,

$$\begin{aligned} T_1 &\leq 1(Y_i \leq z) \sup_p |\hat{Q}(p\hat{F}(z)) - Q(pF(z))| \delta(\delta-1) \int_0^1 (1-p)^{\delta-2} p dp \\ &= 1(Y_i \leq z) \sup_p |\hat{Q}(p\hat{F}(z)) - Q(pF(z))| \end{aligned}$$

which satisfies (ii) of Lemma A5. Similar arguments can be used for the terms T_2 and T_3. Next, $T_4 \leq T_1$ since $1(Y_i < \hat{Q}(p\hat{F}(z)))$ is less than $1(Y_i < z)$ so that T_4 satisfies (ii) of Lemma A5. For T_5 we have that since $Q(pF(z)) \leq z$ for $p \leq 1$ then

$$\begin{aligned} T_5 &\leq z \left| \delta(\delta-1) \int_{\hat{F}(Y_i)/\hat{F}(z)}^{F(Y_i)/F(z)} (1-p)^{\delta-2} dp \right| \\ &\leq z\delta \left| \left(1 - \frac{\hat{F}(Y_i)}{\hat{F}(z)}\right)^{\delta-1} - \left(1 - \frac{F(Y_i)}{F(z)}\right)^{\delta-1} \right| \\ &\leq z\delta \sup_y \left| \left(1 - \frac{\hat{F}(y)}{\hat{F}(z)}\right)^{\delta-1} - \left(1 - \frac{F(y)}{F(z)}\right)^{\delta-1} \right| \\ &= o_p(1) \end{aligned}$$

using the fact that z and δ are fixed and arguments similar to those used in the proof of Proposition S1(B). Therefore the term T_5 satisfies (ii) of Lemma A5. The same argument applies to the final term T_6. **Q.E.D.**

Proof of Proposition S3(A) and S3(B): Because (recalling that $\delta = 1$),

$$E(\frac{1}{z}(z - Y_i).1(Y_i \leq z)) = P^\delta$$

and,

$$\begin{aligned} 0 &\leq \frac{1}{z}(z - Y_i).1(Y_i \leq z) \\ &\leq \frac{1}{z}(z - y_l) \leq 1 \end{aligned}$$

then we have by the Strong Law of Large Numbers that, $\hat{P}^\delta \xrightarrow{a.s.} P^\delta$, so that $\hat{P}^\delta \xrightarrow{p} P^\delta$. Also by the Lindeberg-Levy Central Limit Theorem we have that,

$$\sqrt{N}(\hat{P}^\delta - P^\delta) \xrightarrow{d} N(0, V(\hat{P}^\delta))$$

where,

$$V(\hat{P}^\delta) = E\left(\frac{1}{z^2}(z - Y_i)^2.1(Y_i \leq z)\right) - (P^\delta)^2.$$

Finally the LLN implies that,

$$\frac{1}{Nz^2}\sum_{i=1}^{N}(z - Y_i)^2.1(Y_i \leq z)) \xrightarrow{p} E\left(\frac{1}{z^2}(z - Y_i)^2.1(Y_i \leq z)\right)$$

so that we have,

$$\hat{V}(\hat{P}^\delta) \xrightarrow{p} V(\hat{P}^\delta).$$

Q.E.D.

Proof of Proposition E2(A): The result follows similarly to Proposition S3(A) from Lemma A1, A3 and A4 plus Lemma A2 which gives,

$$\sqrt{N}\left(\left(p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z))\right) - (pG(F(z)) - G(pF(z)))\right) \Rightarrow \Upsilon(p, F(z))$$

Q.E.D.

Proof of Proposition E2(B): This result follows in a similar fashion to the proof of Proposition E1(B) with adjustments similar to those used for the S-gini poverty index in Proposition S2(B).**Q.E.D.**

3 Computation of Influence Curves

In this section we consider the issue of computing the influence curves for the inequality, welfare and poverty indices. Throughout the appendix we use some of the facts discussed in Section 3 of the text regarding the computation of the indices.. Recall that we are using the shorthand $\hat{p}_j = \hat{F}(y_j)$ and that $\hat{\pi}_j = \hat{p}_j - \hat{p}_{j-1}$. Let $\hat{p}_0 = 0$. Then interval $(\hat{p}_{j-1}, \hat{p}_j]$, $\hat{Q}(p) = y_j$. Also on the same interval,

$$\hat{G}(p) = (p - \hat{p}_{j-1})y_j + \sum_{l=1}^{j-1} \hat{\pi}_l y_l = y_j p + a_j$$

where by convention $a_1 = 0$ so that on the interval $(0, \hat{p}_1]$ we have that $\hat{G}(p) = py_1$. As in the calculations performed in Section 4.3 we use the fact that,

$$\int_0^1 = \sum_{j=1}^{\dot{N}} \int_{\hat{p}_{j-1}}^{\hat{p}_j}$$

along with the definitions given above to calculate the estimates of the influence curves for the indices.

3.1 Influence Curves for S-Gini indices

The key component of the influence curve for the S-gini related indices is the term,

$$\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\hat{\phi}_i(p;G)dp = \sum_{l=1}^{4} \hat{I}_l^S$$

Then we are required to compute the following;

$$\begin{aligned}
\hat{I}_1^S &= \delta(\delta-1)\int_0^1 (1-p)^{\delta-2} p\hat{Q}(p)dp \\
\hat{I}_2^S &= -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\hat{G}(p)dp \\
\hat{I}_3^S &= -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\hat{Q}(p)1(Y_i < \hat{Q}(p))dp \\
\hat{I}_4^S &= \delta(\delta-1)\int_0^1 (1-p)^{\delta-2} Y.1(Y_i < \hat{Q}(p))dp
\end{aligned}$$

each of which is considered in turn:

(i)

$$\hat{I}_1^S = \sum_{j=1}^{\dot{N}} y_j \hat{d}_j^1$$

where,

$$\hat{d}_j^1 = -\delta\left(\hat{p}_j\left(1-\hat{p}_j\right)^{\delta-1} - \hat{p}_{j-1}\left(1-\hat{p}_{j-1}\right)^{\delta-1}\right) - \left(\left(1-\hat{p}_j\right)^{\delta} - \left(1-\hat{p}_{j-1}\right)^{\delta}\right)$$

(ii)

$$\hat{I}_2^S = -\sum_{j=1}^{\dot{N}} \left(\hat{d}_j^2 + \hat{d}_j^3 y_j\right)$$

where,

$$\hat{d}_j^2 = -\delta\left(\left(1-\hat{p}_j\right)^{\delta-1}\hat{G}(\hat{p}_j) - \left(1-\hat{p}_{j-1}\right)^{\delta-1}\hat{G}(\hat{p}_{j-1})\right)$$

$$\hat{d}_j^3 = -\left(\left(1-\hat{p}_j\right)^{\delta} - \left(1-\hat{p}_{j-1}\right)^{\delta}\right)$$

with

$$\hat{G}(\hat{p}_j) = \sum_{l=1}^{j} \hat{\pi}_l y_l = \hat{p}_j y_j + a_j = \hat{p}_j y_{j+1} + a_{j+1}$$

(iii)

$$\hat{I}_3^S = -\sum_{j=1}^{\dot{N}} 1(Y_i < y_j) y_j \hat{d}_j^4$$

$$\hat{d}_j^4 = -\delta\left(\left(1-\hat{p}_j\right)^{\delta-1} - \left(1-\hat{p}_{j-1}\right)^{\delta-1}\right)$$

(iv)

$$\hat{I}_4^S = Y_i \sum_{j=1}^{\dot{N}} 1(Y_i < y_j) \hat{d}_j^4$$

Then we can compute the S-gini influence curves using:

$$\begin{aligned}
\hat{\phi}_i(\hat{W}^\delta) &= \sum_{l=1}^{4} \hat{I}_l^S \\
\hat{\phi}_i(\hat{I}_A^\delta) &= (Y_i - \hat{\mu}) - \hat{\phi}_i(W^\delta) \\
\hat{\phi}_i(\hat{I}_R^\delta) &= -\frac{1}{\hat{\mu}}\hat{\phi}_i(W^\delta) - \frac{1}{\hat{\mu}}(Y_i - \hat{\mu})(1 - \hat{I}_R^\delta)
\end{aligned}$$

3.2 Influence curves for E-Gini indices

For the E-gini indices the key component has the form,

$$\frac{1}{\alpha}\int_0^1 \alpha(\mu.p - G(p))^{\alpha-1}\left(p(Y_i-\hat{\mu}) - \hat{\phi}_i(p;G)\right)dp = \frac{1}{\alpha}\sum_{l=1}^{4}\hat{I}_l^E$$

where

$$\begin{aligned}
\hat{I}_1^E &= \int_0^1 \alpha(\hat{\mu}.p - \hat{G}(p))^{\alpha-1}p(Y_i-\hat{\mu})dp \\
\hat{I}_2^E &= -\int_0^1 \alpha(\hat{\mu}.p - \hat{G}(p))^{\alpha-1}p\hat{Q}(p)dp \\
\hat{I}_3^E &= \int_0^1 \alpha(\hat{\mu}.p - \hat{G}(p))^{\alpha-1}\hat{G}(p)dp \\
\hat{I}_4^E &= \int_0^1 \alpha(\hat{\mu}.p - \hat{G}(p))^{\alpha-1}(\hat{Q}(p)-Y_i)1(Y_i<\hat{Q}(p))dp
\end{aligned}$$

(i)

$$\hat{I}_1^E = (Y_i-\hat{\mu})\sum_{j=1}^{\dot{N}}\left\{(\hat{e}_j^1-\hat{e}_j^2)1(\hat{\mu}\neq y_j)+\hat{e}_j^3 1(\hat{\mu}=y_j)\right\}$$

where,

$$\begin{aligned}
\hat{e}_j^1 &= \frac{1}{b_j}\left((b_j\hat{p}_j-a_j)^{\alpha}\hat{p}_j-(b_j\hat{p}_{j-1}-a_j)^{\alpha}\hat{p}_{j-1}\right) \\
\hat{e}_j^2 &= \frac{1}{b_j^2(\alpha+1)}\left((b_j\hat{p}_j-a_j)^{\alpha+1}-(b_j\hat{p}_{j-1}-a_j)^{\alpha+1}\right) \\
\hat{e}_j^3 &= \alpha(-a_j)^{\alpha-1}\left(\frac{\hat{p}_j^2}{2}-\frac{\hat{p}_{j-1}^2}{2}\right)
\end{aligned}$$

with, $b_j=\hat{\mu}-y_j$ and $a_j=\sum_{l=1}^{j-1}\hat{\pi}_l(y_l-y_j)$

(ii)

$$\hat{I}_2^E = -\sum_{j=1}^{\dot{N}} y_j\left\{(\hat{e}_j^1-\hat{e}_j^2)1(\hat{\mu}\neq y_j)+\hat{e}_j^3 1(\hat{\mu}=y_j)\right\}$$

(iii)

$$
\begin{aligned}
\hat{I}_3^E &= \sum_{j=1}^{\dot{N}} y_j \left\{ (\hat{e}_j^1 - \hat{e}_j^2) 1(\hat{\mu} \neq y_j) + \hat{e}_j^3 1(\hat{\mu} = y_j) \right\} \\
&\quad + \sum_{j=1}^{\dot{N}} a_j \left\{ \hat{e}_j^4 1(\hat{\mu} \neq y_j) + \hat{e}_j^5 1(\hat{\mu} = y_j) \right\}
\end{aligned}
$$

where,

$$
\begin{aligned}
\hat{e}_j^4 &= \frac{1}{b_j} \left((b_j \hat{p}_j - a_j)^\alpha - (b_j \hat{p}_{j-1} - a_j)^\alpha \right) \\
\hat{e}_j^5 &= \alpha (-a_j)^{\alpha - 1} (\hat{p}_j - \hat{p}_{j-1})
\end{aligned}
$$

(iv)

$$
\hat{I}_4^E = \sum_{j=1}^{\dot{N}} 1(Y_i < y_j)(y_j - Y_i) \left\{ \hat{e}_j^4 1(\hat{\mu} \neq y_j) + \hat{e}_j^5 1(\hat{\mu} = y_j) \right\}
$$

These four terms can be combined and simplified as follows:

$$
\sum_{l=1}^{4} \hat{I}_l^E = \sum_{j=1}^{\dot{N}} \hat{g}_j^1 \hat{h}_j^1 + \sum_{j=1}^{\dot{N}} \hat{g}_j^2 \hat{h}_j^2
$$

where,

$$
\begin{aligned}
\hat{g}_j^1 &= (\hat{e}_j^1 - \hat{e}_j^2) 1(\hat{\mu} \neq y_j) + \hat{e}_j^3 1(\hat{\mu} = y_j) \\
\hat{g}_j^2 &= \hat{e}_j^4 1(\hat{\mu} \neq y_j) + \hat{e}_j^5 1(\hat{\mu} = y_j) \\
\hat{h}_j^1 &= (Y_i - \hat{\mu}) \\
\hat{h}_j^2 &= (1(Y_i < y_j)(y_j - Y_i) + a_j)
\end{aligned}
$$

Then using these results we have the influence curves for the E-gini indices as:

$$
\begin{aligned}
\hat{\phi}_i(I_A^\alpha) &= \frac{2^\alpha}{\alpha} (\hat{I}_A^\alpha)^{1-\alpha} \sum_{l=1}^{4} \hat{I}_l^E (T_1) \\
\hat{\phi}_i(W^\alpha) &= 2(Y_i - \hat{\mu}) - \hat{\phi}_i(I_A^\alpha) \\
\hat{\phi}_i(I_R^\alpha) &= \frac{1}{\hat{\mu}} \phi_i(I_A^\alpha) - \frac{I_R^\alpha}{\hat{\mu}} (Y_i - \hat{\mu})
\end{aligned}
$$

3.3 Influence Curves for Poverty Indices

As was the case with the calculation of the indices themselves we use a change of variables and the fact that,

$$\int_0^{\hat{F}(z)} = \sum_{j=1}^{N(z)} \int_{\hat{p}_{j-1}}^{\hat{p}_j}$$

For the S-gini related index we must compute,

$$\begin{aligned}
\hat{\phi}_i(P^\delta) &= \hat{\phi}_i(z;F) - \frac{\delta(\delta-1)}{z}\int_0^1 (1-p)^{\delta-2}\left(p\hat{Q}(p\hat{F}(z))\hat{\phi}_i(z;F) + \hat{\phi}_i(p\hat{F}(z);G)\right)dp \\
&= \hat{\phi}_i(z;F) - \frac{\hat{\phi}_i(z;F)}{z}\delta(\delta-1)\int_0^1 (1-p)^{\delta-2} p\hat{Q}(p\hat{F}(z))dp \\
&\quad -\frac{\delta(\delta-1)}{z}\int_0^1 (1-p)^{\delta-2}\hat{\phi}_i(p\hat{F}(z);G)dp \\
&= \hat{\phi}_i(z;F) - \frac{\hat{\phi}_i(z;F)}{z}\tilde{I}_1^S - \frac{1}{z}\sum_{l=2}^{5}\tilde{I}_l^S
\end{aligned}$$

where,

$$\begin{aligned}
\tilde{I}_1^S &= \delta(\delta-1)\int_0^1 (1-p)^{\delta-2} p\hat{Q}(p\hat{F}(z))dp \\
\tilde{I}_2^S &= \delta(\delta-1)\int_0^1 (1-p)^{\delta-2} p\hat{F}(z)Q(p\hat{F}(z))dp \\
\tilde{I}_3^S &= -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2}\hat{G}(p\hat{F}(z))dp \\
\tilde{I}_4^S &= -\delta(\delta-1)\int_0^1 (1-p)^{\delta-2} Q(p\hat{F}(z))1(Y_i < \hat{Q}(p\hat{F}(z)))dp \\
\tilde{I}_5^S &= \delta(\delta-1)\int_0^1 (1-p)^{\delta-2} Y_i.1(Y_i < \hat{Q}(p\hat{F}(z)))dp
\end{aligned}$$

We consider each term in turn and use a change of variables:

(i)

$$\begin{aligned}
\tilde{I}_1^S &= \frac{\delta(\delta-1)}{\hat{F}(z)^\delta}\int_0^{\hat{F}(z)} (\hat{F}(z)-p)^{\delta-2} p\hat{Q}(p)dp \\
&= \frac{1}{\hat{F}(z)^\delta}\sum_{j=1}^{N(z)} y_j \tilde{d}_j^1
\end{aligned}$$

where,

$$\tilde{d}_j^1 = -\delta\left(\hat{p}_j\left(\hat{F}(z)-\hat{p}_j\right)^{\delta-1} - \hat{p}_{j-1}\left(\hat{F}(z)-\hat{p}_{j-1}\right)^{\delta-1}\right) - \left(\left(\hat{F}(z)-\hat{p}_j\right)^{\delta} - \left(\hat{F}(z)-\hat{p}_{j-1}\right)^{\delta}\right)$$

(ii) Similarly,

$$\begin{aligned}\tilde{I}_2^S &= \frac{\hat{F}(z)\delta(\delta-1)}{\hat{F}(z)^{\delta}}\int_0^{\hat{F}(z)}(\hat{F}(z)-p)^{\delta-2}p\hat{Q}(p)dp \\ &= \hat{F}(z)\tilde{I}_1^S\end{aligned}$$

(iii) Next

$$\begin{aligned}\tilde{I}_3^S &= -\frac{\delta(\delta-1)}{\hat{F}(z)^{\delta-1}}\int_0^{\hat{F}(z)}(\hat{F}(z)-p)^{\delta-2}\hat{G}(p)dp \\ &= -\frac{1}{\hat{F}(z)^{\delta-1}}\sum_{j=1}^{N(z)}\left(\tilde{d}_j^2+\tilde{d}_j^3 y_j\right)\end{aligned}$$

where,

$$\tilde{d}_j^2 = -\delta\left(\left(\hat{F}(z)-\hat{p}_j\right)^{\delta-1}\hat{G}(\hat{p}_j)-\left(\hat{F}(z)-\hat{p}_{j-1}\right)^{\delta-1}\hat{G}(\hat{p}_{j-1})\right)$$

$$\tilde{d}_j^3 = -\left(\left(\hat{F}(z)-\hat{p}_j\right)^{\delta}-\left(\hat{F}(z)-\hat{p}_{j-1}\right)^{\delta}\right)$$

with

$$\hat{G}(\hat{p}_j)=\sum_{l=1}^{j-1}\hat{\pi}_l y_l$$

(iv)

$$\tilde{I}_4^S = -\frac{1}{\hat{F}(z)^{\delta-1}}\sum_{j=1}^{N(z)}1(Y_i<y_j)y_j d_j^4$$

where,

$$\tilde{d}_j^4 = -\delta\left(\left(\hat{F}(z)-\hat{p}_j\right)^{\delta-1}-\left(\hat{F}(z)-\hat{p}_{j-1}\right)^{\delta-1}\right)$$

(v)

$$\tilde{I}_5^S = \frac{Y_i}{\hat{F}(z)^{\delta-1}}\sum_{j=1}^{N(z)}1(Y_i<y_j)\tilde{d}_j^4$$

For the E-gini based poverty index the influence curve is given by

$$\begin{aligned}\hat{\phi}_i(P^\alpha) &= -1(Y_i \le z) - \frac{1}{z}\hat{\phi}_i(F(z);G) \\ &\quad + \frac{1}{z}[\hat{T}_E^z]^{\frac{1}{\alpha}-1}\int_0^1 (p\hat{G}(\hat{F}(z)) - \hat{G}(pF(z)))^{\alpha-1}\phi_i(pF(z);\Upsilon)dp \\ &= -1(Y_i \le z) + \tilde{I}_1^E + \frac{1}{z\alpha}[\hat{T}_E^z]^{\frac{1}{\alpha}-1}\sum_{l=2}^{7}\tilde{I}_l^E\end{aligned}$$

where $\hat{T}_E^z$ was already calculated in Section 6.2,

$$\begin{aligned}\tilde{I}_1^E &= -\frac{1}{z}\Big((\hat{F}(z)z - a_{N(z)+1}) + 1(Y_i < z)(z - Y_i)\Big) \\ \tilde{I}_2^E &= z\int_0^1 \alpha(p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z)))^{\alpha-1}pdp \\ \tilde{I}_3^E &= -\int_0^1 \alpha(p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z)))^{\alpha-1}p\hat{Q}(p\hat{F}(z))dp \\ \tilde{I}_4^E &= \Big(\hat{F}(z)(z - \hat{\mu}(z)) - 1(Y_i < z)(z - Y_i)\Big)\int_0^1 \alpha(p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z)))^{\alpha-1}pdp \\ \tilde{I}_5^E &= -\int_0^1 \alpha(p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z)))^{\alpha-1}p\hat{F}(z)\hat{Q}(p\hat{F}(z))dp \\ \tilde{I}_6^E &= \int_0^1 \alpha(p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z)))^{\alpha-1}\hat{G}(p\hat{F}(z))dp \\ \tilde{I}_7^E &= \int_0^1 \alpha(p\hat{G}(\hat{F}(z)) - \hat{G}(p\hat{F}(z)))^{\alpha-1}1(Y_i < \hat{Q}(p\hat{F}(z))(\hat{Q}(p\hat{F}(z)) - Y_i)dp\end{aligned}$$

(i) The term $\tilde{I}_1^E$ has already been given.

(ii)

$$\tilde{I}_2^E = \frac{z}{\hat{F}(z)^2}\sum_{j=1}^{N(z)}\Big\{(\tilde{e}_j^1 - \tilde{e}_j^2)1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^3 1(\hat{\mu}(z) = y_j)\Big\}$$

where,

$$\begin{aligned}\tilde{e}_j^1 &= \frac{1}{b_j}\left(\left(\tilde{b}_j\hat{p}_j - a_j\right)^\alpha \hat{p}_j - \left(\tilde{b}_j\hat{p}_{j-1} - a_j\right)^\alpha \hat{p}_{j-1}\right) \\ \tilde{e}_j^2 &= \frac{1}{b_j^2(\alpha+1)}\left(\left(\tilde{b}_j\hat{p}_j - a_j\right)^{\alpha+1} - \left(\tilde{b}_j\hat{p}_{j-1} - a_j\right)^{\alpha+1}\right) \\ \tilde{e}_j^3 &= \alpha(-a_j)^{\alpha-1}(\frac{\hat{p}_j^2}{2} - \frac{\hat{p}_{j-1}^2}{2})\end{aligned}$$

with, $\tilde{b}_j = \hat{\mu}(z) - y_j$ and $a_j = \sum_{l=1}^{j-1}\hat{\pi}_l(y_l - y_j)$

(iii)

$$\tilde{I}_3^E = -\frac{1}{\hat{F}(z)^2}\sum_{j=1}^{N(z)} y_j \left\{(\tilde{e}_j^1 - \tilde{e}_j^2)1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^3 1(\hat{\mu}(z) = y_j)\right\}$$

(iv)

$$\tilde{I}_4^E = \frac{1}{\hat{F}(z)^2}\left(\hat{F}(z)(z-\hat{\mu}(z)) - 1(Y_i < z)(z - Y_i)\right)\sum_{j=1}^{N(z)} \left\{(\tilde{e}_j^1 - \tilde{e}_j^2)1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^3 1(\hat{\mu}(z) = y_j)\right\}$$

(v)

$$\tilde{I}_5^E = -\frac{\hat{F}(z)}{\hat{F}(z)^2}\sum_{j=1}^{N(z)} y_j \left\{(\tilde{e}_j^1 - \tilde{e}_j^2)1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^3 1(\hat{\mu}(z) = y_j)\right\}$$

(vi)

$$\begin{aligned}\tilde{I}_6^E &= \frac{1}{\hat{F}(z)}\sum_{j=1}^{N(z)} y_j \left\{(\tilde{e}_j^1 - \tilde{e}_j^2)1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^3 1(\hat{\mu}(z) = y_j)\right\} \\ &\quad -\frac{1}{\hat{F}(z)}\sum_{j=1}^{N(z)} a_j \left\{\tilde{e}_j^4 1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^5 1(\hat{\mu}(z) = y_j)\right\}\end{aligned}$$

where,

$$\begin{aligned}\tilde{e}_j^4 &= \frac{1}{\tilde{b}_j}\left(\left(\tilde{b}_j\hat{p}_j - a_j\right)^\alpha - \left(\tilde{b}_j\hat{p}_{j-1} - a_j\right)^\alpha\right) \\ \tilde{e}_j^5 &= \alpha(-a_j)^{\alpha-1}(\hat{p}_j - \hat{p}_{j-1})\end{aligned}$$

(vii)

$$\tilde{I}_7^E = \sum_{j=1}^{N(z)} 1(Y_i < y_j)(y_j - Y_i)\left\{\tilde{e}_j^4 1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^5 1(\hat{\mu}(z) = y_j)\right\}$$

These terms can be combined as follows,

$$\sum_{l=2}^{7} \tilde{I}_l^E = \sum_{j=1}^{N(z)} \tilde{g}_j^1 \tilde{h}_j^1 + \sum_{j=1}^{N(z)} \tilde{g}_j^2 \tilde{h}_j^2$$

where,

$$
\begin{aligned}
\tilde{g}_j^1 &= (\tilde{e}_j^1 - \tilde{e}_j^2)1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^3 1(\hat{\mu}(z) = y_j) \\
\tilde{g}_j^2 &= \tilde{e}_j^4 1(\hat{\mu}(z) \neq y_j) + \tilde{e}_j^5 1(\hat{\mu}(z) = y_j) \\
\tilde{h}_j^1 &= \frac{1}{\hat{F}(z)^2}\Big(z - y_j + \hat{F}(z)(z - \hat{\mu}(z)) - 1(Y_i < z)(z - Y_i)\Big) \\
\tilde{h}_j^2 &= \frac{\hat{F}(z)}{\hat{F}(z)^2}\left(1(Y_i < y_j)(y_j - Y_i) + a_j\right)
\end{aligned}
$$

4 References for Technical Appendix

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